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## A group generated by the Lévy Laplacian

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### 1. INTRODUCTION

The Lévy Laplacian  $\Delta_L$  is one of infinite dimensional Laplacians introduced by P. Lévy in his book [Lé 22]. In his book, he mentioned that  $\Delta_L$  comes from the singular part  $f_s''$  of the second derivative  $f''$ , i.e.,

$$\Delta_L f(x) = \int_0^1 f_s''(x; u) du.$$

This Laplacian has been studied by many authors. In 1975, T. Hida introduced  $\Delta_L$  into the theory of generalized white noise functionals in [Hi 75]. H.-H. Kuo [Ku 83, 89, 92a, 92b] defined the Fourier-Mehler transform on the space  $(\mathcal{S})^*$  of generalized white noise functionals and gave a relation between its transform and  $\Delta_L$ . An interesting characterization of  $\Delta_L$  in terms of rotation groups was obtained by N. Obata [Ob 90]. Recently, T. Hida [Hi 92b] applied  $\Delta_L$  to S. Tomonaga's many time theory in quantum physics.

The purpose of this paper is to construct a group generated by  $\Delta_L$ .

In §2, we will explain a construction of the space of generalized white noise functionals and define the Lévy Laplacian  $\Delta_L^T$  for a finite interval  $T$  in  $\mathbf{R}$  in that space. Moreover, we introduce an operator  $\Delta$  and prove that  $\Delta$  coincides with  $2\Delta_L^T$  on a domain  $D_L^T$  in  $(\mathcal{S})^*$ . In §3, we will construct a  $(C_0)$ -group  $\{G_t\}_{t \in \mathbf{R}}$  generated by  $\Delta_L^T$ . In the last section, we will give a relation between the adjoint operator of Kuo's Fourier-Mehler transform and a group  $\{G_{it}\}_{t \in \mathbf{R}}$ .

### 2. THE LÉVY LAPLACIAN IN THE WHITE NOISE CALCULUS

In this section, we introduce a space of Hida distributions following [Hi 80], [KT 80-82] and [PS 91] (See also, [HKPS 93], [HOS 92] and [Ob 92]) and the Lévy Laplacian defined on a domain in this space.

1) Let  $L^2(\mathbf{R})$  be the Hilbert space of real square-integrable functions on  $\mathbf{R}$  with norm  $|\cdot|_0$ . Consider a Gel'fand triple

$$\mathcal{S} = \mathcal{S}(\mathbf{R}) \subset L^2(\mathbf{R}) \subset \mathcal{S}^* = \mathcal{S}'(\mathbf{R}),$$

where  $\mathcal{S}(\mathbf{R})$  is the Schwartz space consisting of rapidly decreasing functions on  $\mathbf{R}$  and  $\mathcal{S}^*(\mathbf{R})$  is the dual space of  $\mathcal{S}(\mathbf{R})$ .

Let  $A$  be the following operator

$$A = -(d/dx)^2 + x^2 + 1.$$

For each  $p \in \mathbf{Z}$ , we define  $|f|_p = |A^p f|_0$  and let  $\mathcal{S}_p$  be the completion of  $\mathcal{S}$  with respect to the norm  $|\cdot|_p$ . Then the dual space of  $\mathcal{S}'_p$  of  $\mathcal{S}_p$  is the same as  $\mathcal{S}_{-p}$ .

2) Let  $\mu$  be a probability measure on  $\mathcal{S}^*$  with the characteristic functional given by

$$C(\xi) \equiv \int_{\mathcal{S}^*} \exp\{i \langle x, \xi \rangle\} d\mu(x) = \exp\{-\frac{1}{2}|\xi|_0^2\}, \quad \xi \in \mathcal{S}.$$

Let  $(L^2) = L^2(\mathcal{S}^*, \mu)$  be the space of complex-valued square-integrable functionals defined on  $\mathcal{S}^*$  and define the  $S$ -transform by

$$S\varphi(\xi) = C(\xi) \int_{\mathcal{S}^*} \exp\{\langle x, \xi \rangle\} \varphi(x) d\mu(x), \quad \varphi \in (L^2).$$

The Hilbert space admits the well-known Wiener-Itô decomposition:

$$(L^2) = \oplus_{n=0}^{\infty} H_n,$$

where  $H_n$  is the space of multiple Wiener integrals of order  $n \in \mathbf{N}$  and  $H_0 = \mathbf{C}$ . From this decomposition theorem, each  $\varphi \in (L^2)$  is uniquely represented as

$$\varphi = \sum_{n=0}^{\infty} \mathbf{I}_n(f_n), \quad f_n \in L^2_{\mathbf{C}}(\mathbf{R})^{\hat{\otimes} n},$$

where  $\mathbf{I}_n \in H_n$  and  $L^2_{\mathbf{C}}(\mathbf{R})^{\hat{\otimes} n}$  denotes the  $n$ -th symmetric tensor product of the complexification of  $L^2(\mathbf{R})$ .

For each  $p \in \mathbf{Z}, p \geq 0$ , we define the norm  $\|\varphi\|_p$  of  $\varphi = \sum_{n=0}^{\infty} \mathbf{I}_n(f_n)$ , by

$$\|\varphi\|_p = \left( \sum_{n=0}^{\infty} n! |f_n|_{p,n} \right)^{1/2},$$

where  $|\cdot|_{p,n}$  is the norm of  $\mathcal{S}_{\mathbf{C},p}^{\hat{\otimes} n}$  (the  $n$ -th symmetric tensor product of the complexification of  $\mathcal{S}_p$ ). The norm  $\|\cdot\|_0$  is nothing but the  $(L^2)$ -norm. We put

$$(\mathcal{S})_p = \{\varphi \in (L^2); \|\varphi\|_p < \infty\}$$

for  $p \in \mathbf{Z}, p \geq 0$ . Let  $(\mathcal{S})_p^*$  be the dual space of  $(\mathcal{S})_p$ . Then  $(\mathcal{S})_p$  and  $(\mathcal{S})_p^*$  are Hilbert spaces with the norm  $\|\cdot\|_p$  and the dual norm of  $\|\cdot\|_p$ , respectively.

Denote the projective limit space of the  $(\mathcal{S})_p, p \in \mathbf{Z}, p \geq 0$ , and the inductive limit space of the  $(\mathcal{S})_p^*, p \in \mathbf{Z}, p \geq 0$ , by  $(\mathcal{S})$  and  $(\mathcal{S})^*$ , respectively. Then  $(\mathcal{S})$  is a nuclear space and  $(\mathcal{S})^*$  is nothing but the dual space of  $(\mathcal{S})$ . The space  $(\mathcal{S})^*$  is called the space of *Hida distributions* ( or *generalized white noise functionals* ).

Since  $\exp \langle \cdot, \xi \rangle \in (\mathcal{S})$ , the  $S$ -transform is extended to an operator  $U$  defined on  $(\mathcal{S})^*$  :

$$U\Phi(\xi) = C(\xi) \ll \Phi, \exp \langle \cdot, \xi \rangle \gg, \xi \in \mathcal{S},$$

where  $\ll \cdot, \cdot \gg$  is the canonical pairing of  $(\mathcal{S})$  and  $(\mathcal{S})^*$ . We call  $U\Phi$  the  $U$ -functional of  $\Phi$ .

3) We next introduce the definition of the Lévy Laplacian following Kuo [Ku 92] (see also [HKPS 93] ). Let  $U$  be a *Fréchet differentiable* function defined on  $\mathcal{S}$ , i.e. we assume that there exists a map  $U'$  from  $\mathcal{S}$  to  $\mathcal{S}^*$  such that

$$U(\xi + \eta) = U(\xi) + U'(\xi)(\eta) + o(\eta), \eta \in \mathcal{S},$$

where  $o(\eta)$  means that there exists  $p \in \mathbf{Z}, p \geq 0$ , depending on  $\xi$  such that  $o(\eta)/|\eta|_p \rightarrow 0$  as  $|\eta|_p \rightarrow 0$ . Then the first variation

$$\delta U(\xi; \eta) = dU(\xi + \lambda\eta)/d\lambda|_{\lambda=0}$$

is expressed in the form

$$\delta U(\xi; \eta) = \int_{\mathbf{R}} U'(\xi; u) \eta(u) du$$

for every  $\eta \in \mathcal{S}$  by using the generalized function  $U'(\xi; \cdot)$ . We define the *Hida derivative*  $\partial_t \Phi$  of  $\Phi$  to be the generalized white noise functional whose  $U$ -functional is given by  $U'(\xi; t)$ .

**Definition.** (I) A Hida distribution  $\Phi$  is called an  $L$ -functional if for each  $\xi \in \mathcal{S}$ , there exist  $(U\Phi)'(\xi; \cdot) \in L_{loc}^1(\mathbf{R})$ ,  $(U\Phi)''_s(\xi; \cdot) \in L_{loc}^1(\mathbf{R})$  and  $(U\Phi)''_r(\xi; \cdot, \cdot) \in L_{loc}^1(\mathbf{R}^2)$  such that the first and second variations are uniquely expressed in the forms:

$$(U\Phi)'(\xi)(\eta) = \int_{\mathbf{R}} (U\Phi)'(\xi; u) \eta(u) du,$$

and

$$\begin{aligned} (U\Phi)''(\xi)(\eta, \zeta) &= \int_{\mathbf{R}} (U\Phi)''_s(\xi; u) \eta(u) \zeta(u) du \\ &+ \int_{\mathbf{R}^2} (U\Phi)''_r(\xi; u, v) \eta(u) \zeta(v) dudv, \end{aligned} \quad (2.1)$$

for each  $\eta, \zeta \in \mathcal{S}$ , respectively and for any finite interval  $T$ ,  $\int_T (U\Phi)''_s(\cdot; u) du$  is a  $U$ -functional.

(II) Let  $D_L$  denote the set of all  $L$ -functionals. For  $\Phi \in D_L$  and any finite interval  $T$  in  $\mathbf{R}$ , the *Lévy Laplacian*  $\Delta_L^T$  is defined by

$$\Delta_L^T \Phi = U^{-1} \left[ \frac{1}{|T|} \int_T (U\Phi)_s''(\cdot; u) du \right].$$

Remark. Explicit conditions for the uniqueness of the above decomposition (2.1) is given in [HKPS 93, chapter 6].

Let  $T$  be a finite interval in  $\mathbf{R}$ . Take a smooth function  $e$  defined on  $\mathbf{R}$  satisfying  $0 \leq e(u) \leq 1$  for all  $u \in \mathbf{R}$ ,  $e(u) = 1$  for  $|u| \leq 1/2$  and  $e(u) = 0$  for  $|u| \geq 1$ . Let  $\rho_n*$  be the Friedrichs mollifier. Put  $e_n(u) = e(u/n)$  and  $\theta_n^T = \sqrt{2}|\rho_n|_0^{-1}|T|^{-1/2}$ ,  $n = 1, 2, \dots$ . We define an operator  $\Delta$  for a Hida distribution  $\Phi$  by

$$U[\Delta\Phi](\xi) = \lim_{n \rightarrow \infty} \int_{S^*} U\Phi''(\xi)(\theta_n^T e_n(\rho_n * x), \theta_n^T e_n(\rho_n * x)) d\mu(x),$$

if the limit exists in  $U[(S)^*]$ . From now on, we denote  $e_n(\rho_n * x)$  by  $j_n(x)$ . Let  $D_L^T$  denote the set of all  $L$ -functionals  $\Phi$  satisfying  $U\Phi(\eta) = 0$  for  $\eta$  with  $\text{supp}(\eta) \subset T^c$ . In [Sa 94], we obtained the following result. (For the proof, see [Sa 94].)

**THEOREM 1.** *Let  $T$  be a finite interval in  $\mathbf{R}$  and  $\Phi$  an  $L$ -functional in  $D_L^T$ . Then, we have  $\Delta\Phi = 2\Delta_L^T\Phi$ .*

### 3. THE LÉVY LAPLACIAN AS THE INFINITESIMAL GENERATOR

A generalized functional  $\Phi$  is called a *normal functional* if its  $U$ -functional  $U\Phi$  is given by a finite linear combination of

$$\int_{A^k} f(u_1, \dots, u_k) \xi(u_1)^{p_1} \dots \xi(u_k)^{p_k} du_1 \dots du_k, \quad (3.1)$$

where  $f \in L^1(A^k)$ ,  $p_1, \dots, p_k \in \mathbf{N} \cup \{0\}$ ,  $k \in \mathbf{N}$ , and  $A$ : a finite interval in  $\mathbf{R}$ . This functional  $\Phi$  is in  $D_L$ . Let  $\mathcal{N}_T$  denote the set of all normal functionals in  $D_L^T$ . For  $p > 1$  and  $\Phi \in D_L^T$ , we define a  $-p$ -norm  $\|\cdot\|_{-p}$  by

$$\|\Phi\|_{-p}^2 = \sum_{k=0}^{\infty} \|(\Delta_L^T)^k \Phi\|_{-p}^2 \in [0, \infty]$$

and denote the completion of  $\mathcal{N}_T$  with respect to the norm  $\|\cdot\|_{-p}$  by  $D_L^{(-p)}$ . Then  $D_L^{(-p)}$  is the Hilbert space with the norm  $\|\cdot\|_{-p}$  and  $\Delta_L^T$  is a bounded linear operator

on  $D_L^{(-p)}$ . Hence a  $(C_0)$ -group  $\{G_t, t \in \mathbf{R}\}$  is given by

$$G_t = \lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{t^k}{k!} (\Delta_L^T)^k, \quad (3.2)$$

in the sense of the operator norm. It is easily checked that  $\|G_t\| \leq e^{|t|}$ , for any  $t \in \mathbf{R}$ .

Define an operator  $g_t$  on  $\mathcal{N}_T$  for  $t \geq 0$  by

$$U[g_t \Phi](\xi) = \lim_{n \rightarrow \infty} \int_{S^*} U\Phi(\xi + \sqrt{t} \theta_n^T j_n(x)) d\mu(x), \quad \Phi \in \mathcal{N}_T.$$

For a normal functional  $\Phi$  which  $U\Phi$  is given as in (3.1) with the domain  $A^k \subset T^k$ , it is easily checked that

$$U[g_t \Phi](\xi) = \sum_{\nu_1=0}^{[p_1/2]} \cdots \sum_{\nu_k=0}^{[p_k/2]} \frac{p_1! \cdots p_k!}{(2\nu_1)!!(p_1 - 2\nu_1)! \cdots (2\nu_k)!!(p_k - 2\nu_k)!}.$$

$$\left(\frac{2t}{|T|}\right)^{\nu_1 + \cdots + \nu_k} \int_{A^k} f(u_1, \dots, u_k) \xi(u_1)^{p_1 - 2\nu_1} \cdots \xi(u_k)^{p_k - 2\nu_k} du_1 \cdots du_k.$$

Therefore,  $g_t$  is a linear operator from  $\mathcal{N}_T$  to itself. By Theorem 1, it can be checked that  $G_t = g_t$  on  $\mathcal{N}_T$ . Since  $\mathcal{N}_T$  is dense in  $D_L^{(-p)}$ , we have the following

**THEOREM 2.** *For any  $t \geq 0$ ,  $g_t$  is extended to the operator  $G_t$ .*

#### 4. THE FOURIER-MEHLER TRANSFORM AND THE LÉVY LAPLACIAN

An characterization of Hida distributions was obtained by J. Potthoff and L. Streit [PS 91]. From [PS 91], we see that for any  $U$ -functional  $F$ , and  $\xi, \eta$  in  $\mathcal{S}$ , the function  $F(\lambda\xi + \eta)$ ,  $\lambda \in \mathbf{R}$ , extends to an entire function  $F(z\xi + \eta)$ ,  $z \in \mathbf{C}$ . Then we can define an operator  $g_{it}$ ,  $t \in \mathbf{R}$ , by

$$U[g_{it} \Phi](\xi) = \lim_{n \rightarrow \infty} \int_{S^*} U\Phi(\xi + \sqrt{it} \theta_n^T j_n(x)) d\mu(x),$$

if the limit exists. Since  $\mu$  is symmetric, the integral is defined independent of choices of the branch of  $\sqrt{it}$ . As in (3.2), we can naturally define  $G_{it}$ ,  $t \in \mathbf{R}$ , by

$$G_{it} = \lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{(it)^k}{k!} (\Delta_L^T)^k,$$

on  $D_L^{(-p)}$ .

An infinite dimensional Fourier-Mehler transform  $F_\theta$ ,  $\theta \in \mathbf{R}$ , on  $(\mathcal{S})^*$  was defined by H.-H. Kuo [Ku 91] as follows. The transform  $F_\theta \Phi$ ,  $\theta \in \mathbf{R}$  of  $\Phi \in (\mathcal{S})^*$  is defined by the unique generalized white noise functional with the  $U$ -functional

$$U[F_\theta \Phi](\xi) = U\Phi(e^{i\theta}\xi) \exp \left[ \frac{i}{2} e^{i\theta} \sin \theta |\xi|_0^2 \right], \quad \xi \in \mathcal{S}.$$

Moreover, the adjoint operator  $F_\theta^*$  of  $F_\theta$  is given by

$$F_\theta^* \Phi = \sum_{n=0}^{\infty} \mathbf{I}_n(h_n(\Phi; \theta)) \text{ for } \Phi = \sum_{n=0}^{\infty} \mathbf{I}_n(f_n) \in (\mathcal{S}),$$

where

$$h_n(\Phi; \theta) = \sum_{m=0}^{\infty} \frac{(n+2m)!}{n!m!} \left( \frac{i}{2} \sin \theta \right)^m e^{i(m+n)\theta} \tau^{\otimes m} * f_{n+2m};$$

$$\tau^{\otimes m} = \int_{\mathbf{R}^m} \delta_{t_1} \otimes \delta_{t_1} \otimes \cdots \otimes \delta_{t_m} \otimes \delta_{t_m} dt_1 \cdots dt_m.$$

This operator  $F_\theta^*$  is a continuous linear operator on  $(\mathcal{S})$ . (For details, see [Ku 91] and also [HKO 90]) On  $(\mathcal{S})$ , the Gross Laplacian  $\Delta_G$  ( See [Gr 65, 67] ) and the number operator  $N$  is given by

$$\Delta_G \Phi = \int_{\mathbf{R}} \partial_t^2 \Phi dt$$

and

$$N\Phi = \int_{\mathbf{R}} \partial_t^* \partial_t \Phi dt,$$

respectively (see [Ku 86]). The operator  $e^{i\theta N}$  is called the *Fourier- Wiener transform* (see [HKO 90]). Now, we introduce an operator  $e^{\frac{i}{2}\theta \Delta_G}$  from  $(\mathcal{S})$  into itself given by

$$e^{\frac{i}{2}\theta \Delta_G} \Phi = \sum_{n=0}^{\infty} \mathbf{I}_n(\ell_n(\Phi; \theta)); \quad (4.1)$$

$$\ell_n(\Phi; \theta) = \sum_{m=0}^{\infty} \frac{(n+2m)!}{n!m!} \left( \frac{i}{2} \theta \right)^m \tau^{\otimes m} * f_{n+2m},$$

for  $\Phi = \sum_{n=0}^{\infty} \mathbf{I}_n(f_n) \in (\mathcal{S})$ . Then we have the followings.

LEMMA 1.

$$F_\theta^* = e^{i\theta N} \circ e^{\frac{i}{2}(e^{i\theta} \sin \theta) \Delta_G}. \quad (4.2)$$

PROOF: Take  $\Phi = \sum_{n=0}^{\infty} \mathbf{I}_n(f_n) \in (\mathcal{S})$ . From (4.1), we see that

$$e^{\frac{i}{2}(e^{i\theta} \sin \theta) \Delta_G} \Phi = \sum_{n=0}^{\infty} \mathbf{I}_n(\ell_n(\Phi; e^{i\theta} \sin \theta)).$$

Hence,

$$e^{i\theta N} (e^{\frac{i}{2}(e^{i\theta} \sin \theta) \Delta_G} \Phi) = \sum_{n=0}^{\infty} \mathbf{I}_n(e^{in\theta} \ell_n(\Phi; e^{i\theta} \sin \theta)).$$

Since  $e^{in\theta} \ell_n(\Phi; e^{i\theta} \sin \theta) = h_n(\Phi; \theta)$ , we obtain (4.2). ■

LEMMA 2. For any  $\Phi \in (\mathcal{S})$ , we have

$$U[e^{\frac{i}{2}\theta\Delta_G}\Phi](\xi) = \int_{\mathcal{S}^*} U\Phi(\xi + \sqrt{i\theta}y) d\mu(y). \quad (4.3)$$

Remark. For any  $\Phi \in (\mathcal{S})$ ,  $\xi \in \mathcal{S}$  and  $z_1, z_2 \in \mathbb{C}$ , the functional  $U\Phi(z_1\xi + z_2\eta)$ ,  $\eta \in \mathcal{S}$ , can be extended to a functional  $\widetilde{U}\Phi(z_1\xi + z_2y)$ , same symbol  $U\Phi(z_1\xi + z_2y)$ .

PROOF: For  $\Phi = \sum_{n=0}^{\infty} \mathbf{I}_n(f_n) \in (\mathcal{S})$ , the right-hand side of (4.3) has the following expansion:

$$\begin{aligned} & \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} f_n(\mathbf{u}) \int_{\mathcal{S}^*} \{\xi(u_1) + \sqrt{i\theta}x(u_1)\} \cdots \{\xi(u_n) + \sqrt{i\theta}x(u_n)\} d\mu(x) d\mathbf{u} \\ &= \sum_{n=0}^{\infty} \sum_{\nu=0}^{[n/2]} \frac{n!}{(2\nu)!!(n-2\nu)!} (i\theta)^\nu \langle \xi^{\otimes(n-2\nu)}, \tau^\nu * f_n \rangle = \sum_{m=0}^{\infty} \langle \xi^{\otimes m}, \ell_m(\Phi; \theta) \rangle. \end{aligned}$$

From (4.1), we see that the last series is equal to  $U[e^{\frac{i}{2}\theta\Delta_G}\Phi](\xi)$ . ■

Define an operator  $J_n$  by

$$U[J_n\Phi](\xi) = U\Phi \circ j_n(\xi), \quad \Phi \in D_L^{(-p)}, \quad \xi \in \mathcal{S}.$$

For all  $n \in \mathbb{N}$  and  $\Phi \in D_L^{(-p)}$ , we can easily check  $J_n\Phi \in (\mathcal{S})$ . Then we have the following.

THEOREM 3. Let  $\Phi \in D_L^{(-p)}$  be a generalized white noise functional with the  $U$ -functional given by  $\psi(F_1, \dots, F_n)$ , where  $\psi$  is an entire function on  $\mathbb{C}$  and  $F_1, \dots, F_n \in U[\mathcal{N}_T]$ . We assume the condition

$$\sum_{k_1, \dots, k_n=0}^{\infty} \frac{1}{k_1! \cdots k_n!} |\partial_{u_1}^{k_1} \cdots \partial_{u_n}^{k_n} \psi(0, \dots, 0)| < \infty$$

$$\sup_N \int_{\mathcal{S}^*} |((F_1 \circ j_N)^{k_1} \cdots (F_n \circ j_N)^{k_n})(ie^{i\alpha_N(t)}\xi + \sqrt{ie^{i\alpha_N(t)} \sin \alpha_N(t)}x)| d\mu(x) < \infty$$

holds for all  $t > 0$  and  $\xi \in \mathcal{S}$ , where  $\alpha_N(t) = t(\theta_N^T)^2$ . Then

$$\lim_{N \rightarrow \infty} U[\mathbf{F}_{\alpha_N(t)}^* J_N\Phi](\xi) = U[G_{it}\Phi](\xi), \quad \xi \in \mathcal{S}. \quad (4.4)$$

PROOF: From Lemma 2, we have

$$U[e^{\frac{i}{2}e^{i\alpha_N(t)} \sin \alpha_N(t)\Delta_G} J_N\Phi](\xi) = \int_{\mathcal{S}^*} U[J_N\Phi](\xi + \sqrt{ie^{i\alpha_N(t)} \sin \alpha_N(t)}y) d\mu(y).$$



This functional is expressed in the form given by

$$\sum_{\ell=0}^{\infty} \langle \xi^{\otimes \ell}, f_{N,\ell} \rangle,$$

where  $f_{N,\ell} \in \mathcal{S}_C^{\otimes \ell}$ . Hence, from Lemma 1, we get

$$U[\mathbf{F}_{\alpha_N(t)}^* J_N \Phi](\xi) = \sum_{\ell=0}^{\infty} e^{i\alpha_N(t)\ell} \langle \xi^{\otimes \ell}, f_{N,\ell} \rangle.$$

From the condition of this theorem and the Lebesgue convergence theorem, we can calculate as follows:

$$\begin{aligned} \lim_{N \rightarrow \infty} U[\mathbf{F}_{\alpha_N(t)}^* J_N \Phi](\xi) &= \lim_{N \rightarrow \infty} U[e^{\frac{i}{2} e^{i\alpha_N(t)} \sin \alpha_N(t) \Delta_G} J_N \Phi](e^{i\alpha_N(t)} \xi) \\ &= \lim_{N \rightarrow \infty} \int_{S^*} U[J_N \Phi](ie^{i\alpha_N(t)} \xi + \sqrt{ie^{i\alpha_N(t)} \sin \alpha_N(t)} y) d\mu(y) \\ &= \sum_{k_1, \dots, k_n=0}^{\infty} \frac{1}{k_1! \dots k_n!} \partial_{u_1}^{k_1} \dots \partial_{u_n}^{k_n} \psi(0, \dots, 0). \\ \lim_{N \rightarrow \infty} \int_{S^*} ((F_1 \circ j_N)^{k_1} \dots (F_n \circ j_N)^{k_n}) (ie^{i\alpha_N(t)} \xi + \sqrt{ie^{i\alpha_N(t)} \sin \alpha_N(t)} x) d\mu(x). \end{aligned}$$

By the direct calculations, it is easily checked that

$$\begin{aligned} \lim_{N \rightarrow \infty} \int_{S^*} ((F_1 \circ j_N)^{k_1} \dots (F_n \circ j_N)^{k_n}) (ie^{i\alpha_N(t)} \xi + \sqrt{ie^{i\alpha_N(t)} \sin \alpha_N(t)} x) d\mu(x) \\ = U[g_{it} U^{-1}(F_1^{k_1} \dots F_n^{k_n})](\xi) = U[g_{it} U^{-1} F_1](\xi)^{k_1} \dots U[g_{it} U^{-1} F_n](\xi)^{k_n}. \end{aligned}$$

Consequently, we obtain (4.4). ■

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